Equal Temperament Equivalences and Nearest Neighbours in the Tonal Space

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1 Equal Temperament Equivalences

Figure 1 shows a central region of the tonal space with points circled that are equivalent to I in equal-temperament (*ET-equivalent* to I).

$\sharp\sharp I$ −−	$\sharp\sharp V$ −	$\sharp\sharp II$ −	$\sharp\sharp VI$ −	$\sharp\sharp III$ −	$\sharp\sharp VII$	$\sharp\sharp\sharp IV$	$\sharp\sharp\sharp I$	$\sharp\sharp\sharp V$	$\sharp\sharp\sharp II$ +	$\sharp\sharp\sharp VI$ +	$\sharp\sharp\sharp III$ +	$\sharp\sharp\sharp VII$ +
$\sharp VI$ −−	$\sharp III$ −	$\sharp VII$ −	$\sharp\sharp IV$ −	$\sharp\sharp I$ −	$\sharp\sharp V$	$\sharp\sharp II$	$\sharp\sharp VI$	$\sharp\sharp III$	$\sharp\sharp VII$ +	$\sharp\sharp\sharp IV$ +	$\sharp\sharp\sharp I$ +	$\sharp\sharp\sharp V$ +
$\sharp IV$ −−	$\sharp I$ −	$\sharp V$ −	$\sharp II$ −	$\sharp VI$ −	$\sharp III$	$\sharp VII$	$\sharp\sharp IV$	$\sharp\sharp I$	$\sharp\sharp V$ +	$\sharp\sharp II$ +	$\sharp\sharp VI$ +	$\sharp\sharp III$ +
II −−	VI −	III −	VII −	$\sharp IV$ −	$\sharp I$	$\sharp V$	$\sharp II$	$\sharp VI$	$\sharp III$ +	$\sharp VII$ +	$\sharp\sharp IV$ +	$\sharp\sharp I$ +
$\flat VII$ −−	IV −	I −	V −	II −	VI	III	VII	$\sharp IV$	$\sharp I$ +	$\sharp V$ +	$\sharp II$ +	$\sharp VI$ +
$\flat V$ −−	$\flat II$ −	$\flat VI$ −	$\flat III$ −	$\flat VII$ −	IV	I	V	II	VI +	III +	VII +	$\sharp IV$ +
$\flat\flat III$ −−	$\flat\flat VII$ −	$\flat IV$ −	$\flat I$ −	$\flat V$ −	$\flat II$	$\flat VI$	$\flat III$	$\flat VII$	IV +	I +	V +	II +
$\flat\flat I$ −−	$\flat\flat V$ −	$\flat\flat II$ −	$\flat\flat VI$ −	$\flat\flat III$ −	$\flat\flat VII$	$\flat IV$	$\flat I$	$\flat V$	$\flat II$ +	$\flat VI$ +	$\flat III$ +	$\flat VII$ +
$\flat\flat\flat VI$ −−	$\flat\flat\flat III$ −	$\flat\flat\flat VII$ −	$\flat\flat IV$ −	$\flat\flat I$ −	$\flat\flat V$	$\flat\flat II$	$\flat\flat VI$	$\flat\flat III$	$\flat\flat VII$ +	$\flat IV$ +	$\flat I$ +	$\flat V$ +
$\flat\flat\flat IV$ −−	$\flat\flat\flat I$ −	$\flat\flat\flat V$ −	$\flat\flat\flat II$ −	$\flat\flat\flat VI$ −	$\flat\flat\flat III$	$\flat\flat\flat VII$	$\flat\flat IV$	$\flat\flat I$	$\flat\flat V$ +	$\flat\flat II$ +	$\flat\flat VI$ +	$\flat\flat III$ +
$\flat\flat\flat\flat II$ −−	$\flat\flat\flat\flat VI$ −	$\flat\flat\flat\flat III$ −	$\flat\flat\flat\flat VII$ −	$\flat\flat IV$ −	$\flat\flat\flat I$	$\flat\flat\flat V$	$\flat\flat\flat II$	$\flat\flat\flat VI$	$\flat\flat\flat III$ +	$\flat\flat\flat VII$ +	$\flat\flat IV$ +	$\flat\flat I$ +

Figure 1: Central points ET-equivalent to I

2 Choosing a Nearest Neighbour

2.1 Manhattan Distance

The points circled in figure 2 are the points nearest to the central I in each ET equivalence set, simply using Manhattan distance as a metric.

The choice of ♯IV (ET-equivalent to ♭V) is ambiguous between the ♯IV to the right – (2,1) – and the ♭V to the left – (-2,-1).

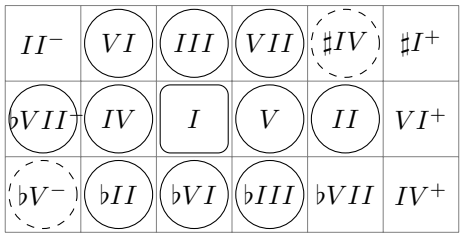

Figure 2: Nearest neighbours to I by Manhattan distance

2.2 Distance from Key Region

The points circled in figure 3 are those nearest to the major key of I in each ET equivalence set. (Square points are within the key.) These could be a better choice of nearest neighbour than those circled above for a tonic function chord on I, or a dominant function chord on V.

The choice of tritone is now unambiguous: the ♭V is most closely connected to the key by (-1,-1) from the IV, whilst the ♯IV is connect by (0,1) to the II and (1,0) to the VII.

However, the choice of ♭VII is now ambiguous. The real ♭VII is connected to the II of the key by a major third – (0,-1) – and the ♭VII⁻ is connected to the IV of the key by a perfect fifth – (-1,0).

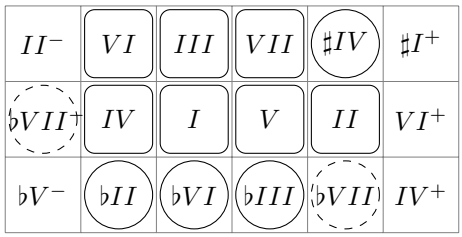

Figure 3: Nearest neighbours to I by distance from the key region

Note that these two policies are not as similar as they look. If we are looking for the nearest neighbour of a dominant chord on V, the choices would be as in the second diagram, since we would assume we're (at least in the very short term) in the tonal region of the tonic resolution. However, the first policy would give the choices shown in figure 4 (since we're now looking for a neighbour to V, not I). A different point will be chosen for a VI, and a different point *may* by chosen for the ♭II and the ♭VII.

2.3 Adopted Strategy

The main circumstances in which we need to choose a nearest neighbour are when picking the point on which a cadence begins after a tonic and when modulating abruptly from one tonic to another. In both cases the first point is a tonic, so the key region is that shown by the squares in figure 3.

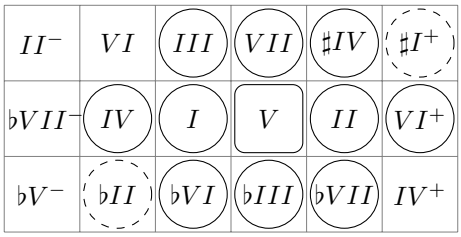

Figure 4: Nearest neighbours to V by Manhattan distance

For the time being, at least, Manhattan distance seems like a reasonably metric. I propose a strategy that arbitrarily picks the ♯IV, rather than the ♭V. This is a genuine ambiguity and in some sense the choice is unimportant, provided a consistent strategy is employed.

This gives the set of points in figure 5 considered to be the closest instances of each ET equivalence set to I.

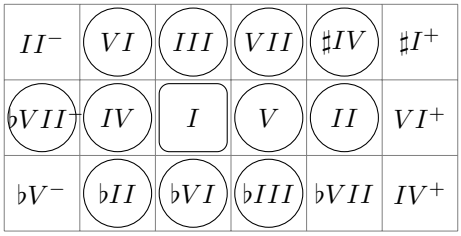

Figure 5: Adopted strategy for picking a closest point to I